SLE-type growth processes and the Yang-Lee singularity

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Abstract

The recently introduced SLE growth processes are based on conformal maps from an open and simply-connected subset of the upper half-plane to the half-plane itself. We generalize this by considering a hierarchy of stochastic evolutions mapping open and simply-connected subsets of smaller and smaller fractions of the upper half-plane to these fractions themselves. The evolutions are all driven by one-dimensional Brownian motion. Ordinary chordal SLE appears at grade one in the hierarchy. At grade two we find a direct correspondence to conformal field theory through the explicit construction of a level-four null vector in a highest-weight module of the Virasoro algebra. This conformal field theory has central charge $c = -22/5$ and is associated to the Yang-Lee singularity. Our construction may thus offer a novel description of this statistical model.

Keywords: Stochastic Löwner evolution, conformal field theory, Yang-Lee singularity.

1 Introduction

A new approach to the description of conformal field theories (CFTs) in two dimensions has recently appeared where instead of discussing objects in terms of local fields and their fusions, one is rather interested in a description based on spatially extended quantities defined through geometry. The differential equations of the stochastic Löwner evolution (SLE) have emerged as a mathematically precise way of describing certain CFTs directly in the continuum, without reference to an underlying lattice.

The chordal SLE processes are constructed through conformal maps from a subset of the upper half-plane onto the half-plane itself. The processes are driven by the random one-dimensional Brownian motion. Properties thereby described have an intrinsic geometrical nature.

The study of these stochastic evolutions or growth processes was initiated by Schramm [1] and has been pursued further in [2, 3, 4, 5, 6, 7, 8, 9, 10], for example. A review for physicists may be found in [11], while [12] contains a mathematical introduction.

An explicit relationship between SLE and CFT has been elucidated recently [13, 14] by considering random walks on the Virasoro group. The link is found through a singular vector at level two in highest-weight modules. The kernel of the vector corresponds to conserved quantities under the random process.

Although the correspondence exists, the number of CFTs having geometrical properties described by SLE is still very limited. Furthermore, there is no apparent pattern assisting in the identification of these new descriptions of field theories.

The aim of the present work is to show that there might be conformal systems described by generalizations of SLE. The approach of Bauer and Bernard [13, 14] may be extended to more general walks than the one generating SLE. A particular class of extensions corresponds to a hierarchy of stochastic evolutions in which SLE appears at grade one. These growth processes are associated to conformal maps of open and simply-connected subsets of smaller and smaller fractions of the upper half-plane onto the fractions themselves, and are all driven by one-dimensional Brownian motion. Using two-sided Brownian motion the stochastic processes may be extended to also describing flows from fractions of the upper half-plane to subsets thereof. At grade two in the hierarchy we find a link to the Yang-Lee singularity through the construction of a level-four null vector. This in turn potentially offers a new geometrical description of that statistical model.

2 Stochastic evolutions

2.1 Stochastic Löwner evolution

Let Y_t be a real-valued continuous function, $t \geq 0$. For each element in the upper half-plane, $z \in \mathbb{H}$, we consider the solution $g_t(z)$ to Löwner's differential equation

$$\partial_t g_t(z) \;=\; \frac{2}{g_t(z) - Y_t}\;, \qquad g_0(z) \;=\; z \tag{1}$$

The factor 2 is conventional but could be changed by re-normalization. Let $\tau = \tau(z)$ denote the time such that the solution $g_t(z)$ exists for all $t \in [0, \tau]$, while for increasing time $\lim_{t\to\tau} g_t(z) = Y_\tau$. Following [1, 8, 12], one may define the evolving hull K_t as the closure of $\{z \in \mathbb{H} : \tau(z) \leq t\}$. In time, it is an increasing sequence of compact sets. As a conformal map from the simply-connected domain $\mathbb{H} \setminus K_t$ onto the open half-plane $\mathbb{H}$, g_t is uniquely determined by the so-called hydrodynamic normalization at infinity:

$$\lim_{z\to\infty} (g_t(z) - z) \;=\; 0 \tag{2}$$

Stochastic Löwner evolutions are growth processes defined by choosing standard one-dimensional (and one-sided) Brownian motion, B_t, as the driving function: $Y_t = \sqrt{\kappa} B_t$, with $B_0 = 0$. The parameter κ characterizes the process which is denoted SLE_κ. For $t, s \geq 0$, the expectation value is normalized as $\mathbf{E}[(\sqrt{\kappa}B_t)(\sqrt{\kappa}B_s)] = \kappa \min(t, s)$.

One defines the function

$$f_t(z) := \; g_t(z) - Y_t \tag{3}$$

It follows that it satisfies the differential equation

$$\partial_t f_t(z) \;=\; \frac{2}{f_t(z)} - \partial_t Y_t\;, \qquad f_0(z) \;=\; z \tag{4}$$

When Y_t denotes Brownian motion its time derivative is thought of as white noise: $dB_t/dt \sim W_t$. The inverse of the function f_t is related to the inverse of the SLE map: $f_t^{-1}(z) = g_t^{-1}(z + Y_t)$. The trace γ of SLE is then defined by

$$\gamma(t) := \; \lim_{z\to 0} f_t^{-1}(z) \tag{5}$$

By construction, z is an element of $\mathbb{H}$, so the limit is taken from the upper half-plane only. The nature of the trace is known to depend radically on κ [8]: for $0 \leq \kappa \leq 4$ it is a simple curve, for $4 < \kappa < 8$ a self-intersecting curve, whereas for $8 < \kappa$ it is space filling. The Hausdorff dimension of the SLE_κ trace is discussed in [8, 15, 16].

2.2 Hierarchy of stochastic evolutions

For positive integer n, we define the open subset of $\mathbb{H}$

$$\mathbb{H}_n \;=\; \{z \in \mathbb{H} : z = re^{i\theta};\; r \in \mathbb{R}_>;\; 0 < \theta < \pi/n\} \tag{6}$$

Note that $\mathbb{H}_1$ is the upper half-plane itself. We now introduce a hierarchy of Löwner-like differential equations whose solutions have properties similar to the SLE maps. For positive integer n we define the differential equation

$$\partial_t g_t(z) \;=\; \frac{2}{g_t^{n-1}(z)\,(g_t^n(z) - Y_t)}\;, \qquad g_0(z) \;=\; z \tag{7}$$

with $Y_0 = 0$. For each $z \in \mathbb{H}_n$ the solution is well-defined up to a time $\tau_n(z)$. Similarly to the ordinary SLE case, the differential equation (7) describes the evolution of the hull $K_t^{(n)}$ defined as the closure of $\{z \in \mathbb{H}_n : \tau_n(z) \leq t\}$.

The solution to (7) is a conformal map from $\mathbb{H}_n \setminus K_t^{(n)}$ onto $\mathbb{H}_n$. To see this, one may generalize the proof of Proposition 2.2 in [12]. One first verifies that $\partial_z g_t(z)$ is well-defined by analyzing $\partial_t\partial_z g_t(z)$. From the evaluation of $\partial_t(g_t(z) - g_t(z'))$ (which is shown to have $(g_t(z) - g_t(z'))$ as a factor), one deduces that $g_t(z) \neq g_t(z')$ when $z \neq z'$. It has thereby been established that g_t is a conformal transformation of $\mathbb{H}_n \setminus K_t^{(n)}$. To show that $g_t(\mathbb{H}_n \setminus K_t^{(n)}) = \mathbb{H}_n$, one studies the inverse flow $h_t(w)$, $w \in \mathbb{H}_n$, which is a solution to

$$\partial_t h_t(w) \;=\; -\frac{2}{h_t^{n-1}(w)(h_t^n(w) - Y_{t_0-t})}\;, \qquad h_0(w) \;=\; w \tag{8}$$

for some $t_0 \geq 0$. The solution $h_t(w)$ is well-defined for $0 \leq t \leq t_0$ since $\partial_t \mathrm{Im}(h_t^n(w)) > 0$ and $|h_t^{n-1}| \geq \min\{|h_t^n(w)|, 1\}$. This ensures that the solution cannot hit the singularities. With $z = h_{t_0}(w)$, $g_t(z) = h_{t_0-t}(w)$ is seen to be a solution to (7) (implying that $h_t(w)$ is indeed the inverse flow), and $g_{t_0}(z) = w$ showing that $w \in g_{t_0}(\mathbb{H}_n \setminus K_t^{(n)})$.

The solution to (7) is determined uniquely by the hydrodynamic normalization at infinity (2). It has the power series expansion

$$g_t(z) \;=\; z + \frac{2t}{z^{2n-1}} + \mathcal{O}(1/|z|^{(2n)})\;, \qquad z \to \infty \tag{9}$$

We refer to the process as being of grade n.

When $n = 1$ (and $Y_t = \sqrt{\kappa}B_t$) we recover the ordinary SLE equation (1). In a subsequent section we shall focus on grade $n = 2$ as it is in this case we find a new relation to CFT and the Yang-Lee singularity.

Two important properties of ordinary SLE are scale invariance and a sort of stationarity. These apply to solutions to (7) as well. In the spirit of Proposition 2.1 in [8] (see also [3]), we have that the growth process defined by (7) is scale invariant in the following sense. For $\alpha > 0$ the process $t \mapsto \alpha^{-1/(2n)} K_{\alpha t}^{(n)}$ has the same law as $t \mapsto K_t^{(n)}$, while the process $(z, t) \mapsto \alpha^{-1/(2n)} g_{\alpha t}(\alpha^{1/(2n)} z)$ has the same law as $(z, t) \mapsto g_t(z)$. Also, the map

$\tilde{g}(z) := (g_{t_1} \circ g_{t_0}^{-1})(z + Y_{t_0}) - Y_{t_0}$ has the same law as $g_{t_1 - t_0}$ when $t_1 > t_0 > 0$. Moreover, $\tilde{g}$ is independent of g_{t_0}. These assertions can be proved by a simple adaptation of the proof for ordinary SLE.

We define f_t through

$$f_t^n(z) \;=\; g_t^n(z) - Y_t \tag{10}$$

It follows that $f_t(z)$ satisfies the differential equation

$$\partial_t f_t(z) \;=\; \frac{2}{f_t^{2n-1}(z)} - \frac{1/n}{f_t^{n-1}(z)} \partial_t Y_t \,, \qquad f_0(z) \;=\; z \tag{11}$$

with a canonical choice of boundary condition. The solution respects the hydrodynamic normalization at infinity (2), and it corresponds to choosing the 'principal root' in the relation (10). As in ordinary SLE, we use f_t to define an SLE-type trace for the hierarchy of evolutions:

$$\gamma_n(t) := \lim_{z \to 0} f_t^{-1}(z) \tag{12}$$

To illustrate our construction, we now consider the situation where the driving function vanishes for all t: $Y_t \equiv 0$ (corresponding to $\kappa \equiv 0$). The differential equation becomes

$$\partial_t g_t(z) \;=\; \frac{2}{g_t^{2n-1}(z)} \,, \qquad g_0(z) \;=\; z \tag{13}$$

with solution

$$g_t(z) \;=\; \left(z^{2n} + 4nt\right)^{1/(2n)} \tag{14}$$

The trace reads

$$\gamma_n(t) \;=\; |(4nt)^{1/(2n)}| e^{i\pi/(2n)} \tag{15}$$

while the hull is

$$K_t^{(n)} \;=\; \{ r e^{i\pi/(2n)} : r \in [0, |(4nt)^{1/(2n)}|] \} \tag{16}$$

Following [8] on ordinary SLE, we may take B to be two-sided Brownian motion (or more generally, Y to be defined for negative t as well). The equation (7) can then also be solved for negative t, in which case g_t is a conformal map from $\mathbb{H}_n$ into a subset of $\mathbb{H}_n$. Indeed, Lemma 3.1 in [8] extends to our case. In the extended version it states that the map $z \mapsto g_{-t}(z)$ has the same distribution as the mapping of z into the principal nth root of $\left(\left(g_t^{-1}((z^n + Y_t)^{1/n})\right)^n - Y_t\right)$. To see this we first observe that for $z \in \mathbb{H}_n$ the principal nth root of $(z^n + x)$ for x real also lies in $\mathbb{H}_n$. For $t_1 \in \mathbb{R}$ we then define the function $\hat{g}_t^{(t_1)}$ as the principal root in the functional relation

$$\left(\hat{g}_t^{(t_1)}(z)\right)^n \;=\; \left(g_{t_1+t} \circ g_{t_1}^{-1}((z^n + Y_{t_1})^{1/n})\right)^n - Y_{t_1} \tag{17}$$

It follows that $\hat{g}_t^{(t_1)}(z)$ is a solution to

$$\partial_t \hat{g}_t^{(t_1)}(z) \;=\; \frac{2}{\left((\hat{g}_t^{(t_1)}(z)\right)^{n-1} \left(\left((\hat{g}_t^{(t_1)}(z)\right)^n - (Y_{t_1+t} - Y_{t_1})\right)} \,, \qquad \hat{g}_0^{(t_1)}(z) \;=\; z \tag{18}$$

We note that $\hat{Y}_t^{(t_1)} := Y_{t_1+t} - Y_{t_1}$ has the same law as Y_t as maps from $\mathbb{R}$ to $\mathbb{R}$, and since

$$\left(\hat{g}_{-t_1}^{(t_1)}(z)\right)^n \;=\; \left(g_{t_1}^{-1}((z^n + Y_{t_1})^{1/n})\right)^n - Y_{t_1} \tag{19}$$

the assertion of the extended lemma follows.

With two-sided Brownian motion at hand, we may define alternatively to (10)

$$f_t^n(z) \;=\; g_{-t}^n(z) - Y_{-t} \tag{20}$$

satisfying

$$\partial_t f_t(z) \;=\; \frac{-2}{f_t^{2n-1}(z)} - \frac{1/n}{f_t^{n-1}(z)}\partial_t Y_{-t}\ , \qquad f_0(z) \;=\; z \tag{21}$$

Choosing the driving function as $Y_t = \sqrt{\kappa} B_{-t}$ we have

$$df_t(z) \;=\; \frac{-2}{f_t^{2n-1}(z)}dt - \frac{\sqrt{\kappa}/n}{f_t^{n-1}(z)}dB_t\ , \qquad f_0(z) \;=\; z \tag{22}$$

This will appear in the link to the Yang-Lee singularity addressed below.

It is remarked that our construction may be interpreted as chordal SLE in $\mathbb{H}_n$. To appreciate this, we introduce $\phi_n(z) = z^n$ and let $G_t^{(n)}(z)$ denote the map in (7) when $Y_t = \sqrt{\kappa} B_t$. $G_t^{(1)}(z)$ thus corresponds to chordal SLE in the upper half-plane $\mathbb{H}_1 = \mathbb{H}$. We then have that

$$\hat{G}_t^{(n)}(z) \;=\; \phi_n^{-1} \circ G_{nt}^{(1)} \circ \phi_n(z) \tag{23}$$

(where ϕ_n^{-1} singles out the principal root) satisfies (7) albeit with $\hat{\kappa} = n\kappa$. As a consequence, $K_t^{(n)}$ is seen to correspond to $\phi_n^{-1}(K_{nt}^{(1)})$.

3 Relation to conformal field theory

3.1 Ordinary SLE

Bauer and Bernard [13, 14] have recently discussed a direct relationship between SLE$_\kappa$ and CFT. Their construction starts from a random walk on the (somewhat formal) Virasoro group:

$$G_t^{-1} dG_t \;=\; -2L_{-2}dt + \sqrt{\kappa} L_{-1} \circ dB_t\ , \qquad G_0 = 1 \tag{24}$$

here written in the Stratonovich interpretation. We shall rather discuss it in the Ito form where it reads

$$G_t^{-1} dG_t \;=\; \left(-2L_{-2} + \frac{\kappa}{2}L_{-1}^2\right)dt + \sqrt{\kappa} L_{-1} dB_t\ , \qquad G_0 = 1 \tag{25}$$

G_t is an element of Vir_- obtained by exponentiating the negative modes, L_n, $n < 0$, of the Virasoro algebra. We write a generic element $G \in Vir_-$ as

$$G \;=\; \cdots e^{x_2 L_{-2}} e^{x_1 L_{-1}} \tag{26}$$

and recall the definition of the Virasoro algebra:

$$[L_n, L_m] \;=\; (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \tag{27}$$

The central charge c plays a prominent role in CFT. As we shall discuss, it is through the construction of singular vectors in highest-weight modules of the Virasoro algebra that the connection to SLE_κ is established [13, 14].

The conformal transformation generated by (25) acts on a primary field of weight Δ as

$$G_t^{-1}\phi_\Delta(z)G_t \;=\; \left(\partial_z f_t(z)\right)^\Delta \phi_\Delta(f_t(z)) \tag{28}$$

for some conformal map f_t to be determined[1]. Using that the Virasoro generators act as

$$[L_n, \phi_\Delta(z)] \;=\; (z^{n+1}\partial_z + \Delta(n+1)z^n)\phi_\Delta(z) \tag{29}$$

one finds that the conformal map associated to the random process (25) must be a solution to the stochastic differential equation

$$df_t(z) \;=\; \frac{2}{f_t(z)}dt - \sqrt{\kappa}dB_t\ , \qquad f_0(z) \;=\; z \tag{30}$$

corresponding to (4). This follows from computing the Ito differential of (28) and is discussed in more details in [13, 14].

Observables of the random process (25) are thought of as functions on the Virasoro group, $F(G_t)$, and a goal is to find the evolution for the expectation values of these observables. With the left-invariant vector fields, ∇_ℓ, defined by

$$\nabla_\ell F(G_t) \;=\; \frac{d}{du}F(G_t e^{uL_\ell})|_{u=0} \tag{31}$$

one has [13] that the expectation value $\mathbf{E}[F(G_t)]$ satisfies

$$\partial_t \mathbf{E}[F(G_t)] \;=\; \mathbf{E}[\left(-2\nabla_{-2} + \frac{\kappa}{2}\nabla_{-1}^2\right)F(G_t)] \tag{32}$$

We shall be interested in observables of the form $F_\Delta(G_t) = G_t|\Delta\rangle$ where $|\Delta\rangle$ is the highest-weight vector of weight Δ in the Verma module $\mathcal{V}_\Delta = Vir_-|\Delta\rangle$ (see [13, 14] and below). In this case the expectation value reads

$$\partial_t \mathbf{E}[G_t|\Delta\rangle] \;=\; \mathbf{E}[G_t\left(-2L_{-2} + \frac{\kappa}{2}L_{-1}^2\right)|\Delta\rangle] \tag{33}$$

For some values of κ (in relation to the central charge c), the linear combination $-2L_{-2} + \frac{\kappa}{2}L_{-1}^2$ will produce a singular vector when acting on the highest-weight vector in a highest-weight module. This is an important point as it enables one, through the representation theory of the algebra, to find quantities conserved in mean under the random process.

[1] For simplicity, we do not distinguish explicitly between boundary and bulk primary fields, nor do we write the anti-holomorphic part.

The Verma module $\mathcal{V}_\Delta$ contains the singular vector at level two

$$|\Delta;2\rangle \;=\; \left(L_{-2}-\frac{\kappa}{4}L_{-1}^2\right)|\Delta\rangle \tag{34}$$

provided $L_1|\Delta;2\rangle \;=\; L_2|\Delta;2\rangle \;=\; 0$. It is straightforward to show that this implies the parameterizations

$$c_\kappa \;=\; 1-\frac{3(4-\kappa)^2}{2\kappa}\;, \qquad \Delta_\kappa \;=\; \frac{6-\kappa}{2\kappa} \tag{35}$$

The expectation value of the observable $F_\Delta(G_t)=G_t|\Delta\rangle$ thus vanishes (33):

$$\partial_t\mathbf{E}[G_t|\Delta\rangle] \;=\; 0 \tag{36}$$

We see that this direct relationship between SLE$_\kappa$ evolutions and CFT is through the existence of a level-two singular vector in a highest-weight module. As discussed in [14], this relationship provides links between conformal correlation functions and probabilities in SLE$_\kappa$.

3.2 Extended SLE

Since Brownian motion played a significant role in the derivation of (32) and (33), and hence in the correspondence between SLE and CFT, it remains unclear how to treat more general random processes than (25). An extension invites itself, though. Namely, consider the random walk

$$G_t^{-1}dG_t \;=\; v_{-n}L_{-2n}dt+\sqrt{\kappa}u_{-n}L_{-n}\circ dB_t\;, \qquad G_0 \;=\; 1 \tag{37}$$

or in the Ito interpretation

$$G_t^{-1}dG_t \;=\; \left(v_{-n}L_{-2n}+\frac{\kappa u_{-n}^2}{2}L_{-n}^2\right)dt+\sqrt{\kappa}u_{-n}L_{-n}dB_t\;, \qquad G_0 \;=\; 1 \tag{38}$$

In this case we have

$$\partial_t\mathbf{E}[F(G_t)] \;=\; \mathbf{E}[\left(v_{-n}\nabla_{-2n}+\frac{\kappa u_{-n}^2}{2}\nabla_{-n}^2\right)F(G_t)] \tag{39}$$

and in particular

$$\partial_t\mathbf{E}[G_t|\Delta\rangle] \;=\; \mathbf{E}[G_t\left(v_{-n}L_{-2n}+\frac{\kappa u_{-n}^2}{2}L_{-n}^2\right)|\Delta\rangle] \tag{40}$$

To relate this to the SLE-type differential equations discussed above, we write (11) and (22) uniformly as

$$df_t(z) \;=\; \frac{2(-1)^{s+1}}{(f_t(z))^{2n-1}}dt-\frac{\sqrt{\kappa}/n}{(f_t(z))^{n-1}}dB_t \tag{41}$$

where $s = 1$ or $s = 2$ depending on the choice of relation (10) or (20), respectively. Taking the Ito differential of the right hand side of (28) results in

$$\begin{aligned} & d\{(\partial_z f_t(z))^\Delta \phi_\Delta(f_t(z))\} \\ = & \left[\left(2(-1)^{s+1} + \frac{\kappa(n-1)}{2n^2}\right) L_{-2n}dt - \frac{\sqrt{\kappa}}{n} L_{-n} dB_t, (\partial_z f_t(z))^\Delta \phi_\Delta(f_t(z))\right] \\ & + \frac{\kappa}{2n^2}\left[L_{-n}, \left[L_{-n}, (\partial_z f_t(z))^\Delta \phi_\Delta(f_t(z))\right]\right] dt \end{aligned} \tag{42}$$

while the Ito differential of the left hand side of (28) generated by the random walk (38) reads

$$\begin{aligned} d\{G_t^{-1}\phi_\Delta(f_t(z))G_t\} &= \left[-v_{-n}L_{-2n}dt - \sqrt{\kappa}u_{-n}L_{-n}dB_t, G_t^{-1}\phi_\Delta(f_t(z))G_t\right] \\ &\quad + \frac{\kappa u_{-n}^2}{2}\left[L_{-n}, \left[L_{-n}, G_t^{-1}\phi_\Delta(f_t(z))G_t\right]\right] dt \end{aligned} \tag{43}$$

A comparison of the two Ito differentials suggests considering the walk

$$G_t^{-1}dG_t = \left(\left(2(-1)^s - \frac{\kappa(n-1)}{2n^2}\right) L_{-2n} + \frac{\kappa}{2n^2}L_{-n}^2\right) dt + \frac{\sqrt{\kappa}}{n} L_{-n} dB_t \tag{44}$$

According to this, we should be looking for singular vectors of the form

$$|\Delta; 2n\rangle = \left(\left(2(-1)^s - \frac{\kappa(n-1)}{2n^2}\right) L_{-2n} + \frac{\kappa}{2n^2}L_{-n}^2\right) |\Delta\rangle \tag{45}$$

The upset, however, is that for $n > 1$

$$\begin{aligned} L_1|\Delta; 2n\rangle &= \left(\left((2n+1)\left(2(-1)^s - \frac{\kappa(n-1)}{2n^2}\right) + \frac{(n+1)\kappa}{2n^2}\right) L_{-(2n-1)} \right. \\ &\quad \left. + \frac{(n+1)\kappa}{n^2} L_{-n}L_{-(n-1)}\right) |\Delta\rangle \\ &\neq 0 \end{aligned} \tag{46}$$

for all κ. This means that (45) can be a singular vector only when $n = 1$, and is then given by (34) (when $s = 1$). One should not be completely discouraged by this. The pivotal property of the state $\left(L_{-2} - \frac{\kappa}{4}L_{-1}^2\right)|\Delta\rangle$ appearing in (33) and applications thereof [14], is that it vanishes in the quotient space of $\mathcal{V}_\Delta$ where all singular vectors have been factored out. In other words, it is a null vector. This means that we do not have to insist that the vector (45) is a (primitive) singular vector itself, but only require that it is a linear combination of descendants of (primitive) singular vectors. An example is provided below.

4 Yang-Lee singularity

Generically, the Verma module $\mathcal{V}_\Delta$ is irreducible. Minimal models [17, 18] are examples of CFTs for which it is reducible. They are labeled by a pair of positive and co-prime integers $p > p'$, and are denoted $\mathcal{M}(p, p')$. The central charge is

$$c \;=\; 1 - 6\frac{(p-p')^2}{pp'} \tag{47}$$

while the spectrum of primary fields or highest-weight representations have conformal weights

$$\Delta_{r,s} \;=\; \frac{(rp - sp')^2 - (p-p')^2}{4pp'}\ , \qquad 1 \leq r < p'\ , \qquad 1 \leq s < p \tag{48}$$

with $\Delta_{p'-r,p-s} = \Delta_{r,s}$. There are two singular vectors not being descendants of singular vectors themselves, and they appear at levels rs and $(p'-r)(p-s)$.

For $p' = 2$, there is only one primary field admitting a singular vector at level two, in which case $(r, s) = (1, 2)$. For $p' > 2$, on the other hand, there are two such fields, labeled by $(1, 2)$ and $(2, 1)$, respectively. It is easily verified that

$$\Delta_{1,2} \;=\; \Delta_{\kappa=4p/p'}, \qquad \Delta_{2,1} \;=\; \Delta_{\kappa=4p'/p} \tag{49}$$

It follows that SLE_κ and $\mathrm{SLE}_{16/\kappa}$, with $\kappa = 4p/p'$, may be linked to the same minimal model, albeit via two different primary fields in the model.

The simplest example of a null vector of the form (45) for $n > 1$ that we have found is a level-four vector in the minimal model $\mathcal{M}(5, 2)$ with $c = -22/5$, cf. (47). This model offers a CFT description of the statistical Yang-Lee singularity. Unlike ordinary SLE (except $\mathrm{SLE}_{\kappa=6}$ which is known to correspond to percolation, and has central charge $c = 0$), the null vector appears in the *identity module* having singular vectors

$$\begin{aligned} |0\rangle_1 \;&=\; L_{-1}|0\rangle \\ |0\rangle_4 \;&=\; \left(L_{-4} + \frac{5}{27}L_{-3}L_{-1} - \frac{5}{3}L_{-2}^2 + \frac{125}{27}L_{-2}L_{-1}^2 - \frac{125}{108}L_{-1}^4\right)|0\rangle \end{aligned} \tag{50}$$

The null vector of our interest reads

$$\begin{aligned} |0;4\rangle \;&=\; |0\rangle_4 + \left(-\frac{5}{27}L_{-3} - \frac{125}{27}L_{-2}L_{-1} + \frac{125}{108}L_{-1}^3\right)|0\rangle_1 \\ &=\; \left(L_{-4} - \frac{5}{3}L_{-2}^2\right)|0\rangle \end{aligned} \tag{51}$$

Comparing this to (45), we find that the Yang-Lee singularity is related to a grade-two SLE-type evolution with

$$\kappa \;=\; 40 \tag{52}$$

and $s = 2$. There is no non-negative solution for κ when $s = 1$. In summary, this SLE-type evolution reads

$$\partial_t g_t(z) \ = \ \frac{2}{g_t(z)\left(g_t^2(z) - \sqrt{40}B_t\right)} \ , \qquad g_0(z) \ = \ z \tag{53}$$

or in terms of $f_t(z)$, cf. (22) and (41):

$$df_t(z) \ = \ \frac{-2}{f_t^3(z)}dt - \frac{\sqrt{10}}{f_t(z)}dB_t \ , \qquad f_0(z) \ = \ z \tag{54}$$

This provides a novel approach to the Yang-Lee model, and may eventually lead to an explicit geometric realization.

5 Conclusion

As observed by Bauer and Bernard [13, 14], SLE may be linked to CFT through the construction of a singular vector in a Virasoro highest-weight module. We have extended their approach, and found that the Yang-Lee singularity may be described by a generalization of the SLE differential equation. This new stochastic evolution appears at grade two in a hierarchy of SLE-type growth processes in which ordinary SLE appears at grade one. Their approach has recently been extended also to stochastic evolutions in superspace and superconformal field theory [19]. Another extension will appear elsewhere where it is discussed how SLE-type growth processes may be linked to CFT via (non-primary) descendant fields. A possible classification of these links will also be addressed.

Acknowledgements

The authors thank L.-P. Arguin, P. Jacob, F. Loranger, P. Mathieu, and Y. Saint-Aubin for discussions, and JR thanks M. Bauer, G. Lawler and O. Schramm for helpful correspondence. The authors are very grateful to S. Chakravarty for pointing out a mistake in a previous version of this work.

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